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Wilson Renormalization Group formulation of Real Time thermal field theories

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Abstract

We apply Renormalization Group techniques to the Real Time formulation of thermal field theory. Due to the separation between the $T = 0$ and the $T \neq 0$ parts of the propagator in this formalism, one can derive exact evolution equations for the Green functions describing the effect of integrating out thermal fluctuations of increasing wavelengths, the initial conditions being the renormalized Green functions of the $T = 0$ theory. As a first application, we study the phase transition for the real scalar theory, computing the order of the transition, the critical temperature, and critical exponents, in different approximations to the evolution equations for the scalar potential.

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1 Introduction

The dynamics of a second order, or weakly first order, phase transition is governed by long wavelength fluctuations of the order parameter. These fluctuations, whose length scale is much larger than the inverse temperature of the system, are essentially classical, since the probability of a quantum fluctuation over such scales is highly suppressed. For this reason, the details of the microscopic theory are not relevant to the description of the critical behaviour of the system, which can be successfully studied by using classical models, such as the Ising model for ferromagnets or the Ginsburg-Landau theory for superconductors.

In these models, a free energy can be defined as a functional of a macroscopic order parameter, which is allowed to vary only on large length scales. The effect of the short wavelength (quantum and thermal) fluctuations is incorporated by the parameters appearing in the free energy (masses, coupling constants, etc.), which are usually treated as phenomenological parameters but should in principle be computable, starting from the underlying theory.

The Wilson Renormalization Group (RG) [1] provides the natural framework in which this procedure can be systematically carried out. The main idea is to start from a microscopic theory, the parameters of which are supposed to be known, and then progressively integrate out the high frequency modes of the order parameter down to some infra-red cutoff Λ. In this way one obtains a coarse-grained order parameter and the corresponding effective action, which can be used as the relevant tool to describe the system.

In this paper, we will apply this idea to quantum field theory at finite temperature. Our approach will be the following. First, we will assume that we "know" the zero-temperature renormalized quantum field theory, which means that there exists some reliable approximation method (perturbation theory, lattice simulations, etc.) to compute the zero-temperature renormalized Green functions. In this way, the renormalization constants can be fixed by the experimental measurements and all the parameters of the theory are known. In a RG language, we assume that all the quantum fluctuations have already been integrated out.

Second, we will integrate out thermal modes only, for frequencies higher than the infrared cutoff Λ. For non-vanishing values for Λ, we will obtain coarse-grained order parameter and free energy, which are the appropriate objects to study the dynamics of long wavelength thermal fluctuations. In the limit $\Lambda \to 0$, we will obtain the finite temperature quantum field theory in thermal equilibrium.

In this approach, since all the quantum fluctuations are integrated out from the beginning, it will be possible to relate the thermal field theory and the physical (renormalized) quantum field theory at zero temperature in a transparent way.

The ideal framework to perform this program of coarse-graining of thermal fluctuations is the Real Time (RT) formulation of thermal field theories [2], in which the thermal part in the free propagators is well separated from the zero-temperature quantistic one. As is well known, the price to pay for this is a doubling of the number of degrees of freedom. These "ghost" fields are necessary to cancel the so-called pinch singularities, which are due to the

fact that the thermal fluctuations are on-shell. Because of these technical complications Matsubara's Imaginary Time (IT) approach [3] is more popular in the literature.

In particular, the coarse-graining procedure can be precisely formulated in the so-called Closed-Time-Path (CTP) formalism [4] (see refs. [5, 6] for a clear presentation). Indeed, the description of a system in which only the short wavelength fluctuations are in thermal equilibrium can be achieved by modifying the density matrix with respect to the thermal one, and the CTP was designed just to describe systems with a generic density matrix.

Anyway, the reader not familiar with the CTP formalism should not worry too much about it. As we will discuss in Appendix A, the modification of the density matrix that we will consider is equivalent to working in the more familiar RT formulation of Niemi and Semenoff [2, 7] with a modified distribution function, given by the Bose-Einstein distribution function (or the Fermi-Dirac, for fermions) multiplied by the cutoff function.

As a first stage of this program, in this paper we illustrate our method by considering the well-studied self-interacting real scalar theory. As is well known, this model belongs to the universality class of the Ising model, and then it has a second order phase transition. However, perturbation theory fails to reproduce this result even after the resummation of daisy and super-daisy diagrams [8], unless the gap equations are solved to $O(\lambda^2)$, λ being the quartic coupling constant [9].

We derive the "exact" Wilson RG equation for the effective potential and approximate it by expanding in powers of space-time derivatives of the field. As we will discuss, even after our approximations, the effective potential computed in this approach includes more contributions than the super-daisy resummed one in [8] and the prediction of a second order phase transition will emerge in a clear way. In particular, we will see that the running of the coupling constant, which is neglected in perturbation theory, has a very strong effect. At the critical temperature, the effective coupling constant vanishes together with the thermal mass, and this cures the severe infra-red problems encountered in perturbation theory.

The paper is organized as follows. In Section 2 we define the cutoff thermal field theory and discuss its relationship with other RG approaches. In Section 3 we derive the RG flow equations for the quantities of interest. In Section 4 the evolution equation for the cutoff effective potential is approximated in various ways in order to perform the numerical analysis. Section 5 contains the results of our numerical study and Section 6 some conclusions and further comments. In Appendix A we briefly describe the Closed Time Path method and how it applies to our case. Appendix B contains a discussion of the problem of pinch singularities appearing in the kernel of RG flow equations.

2 Cutoff thermal field theory

As explained in the introduction, our aim is to define a coarse-graining procedure in which all the quantum fluctuations, and the high frequency thermal ones down to an infra-red cutoff Λ, are integrated out. In this section we will construct a path integral representation of the resulting generating functional of coarse-grained Green functions, in the case of a self-interacting real scalar field.

2.1 Cutoff propagators

Let us consider a free field, expanded in the usual manner in terms of annihilation and creation operators

$$\hat{\phi}(x) = \int \frac{d^3k}{(2\pi)^3\, 2\omega_k} \left[a_k \exp(-ik\cdot x) + a_k^\dagger \exp(ik\cdot x) \right], \tag{1}$$

with $k_0 = \omega_k = \sqrt{\vec{k}^2 + m^2}$; m^2 represents here the zero-temperature, physical mass of the scalar particle.

In the case of an ideal gas in thermal equilibrium at a temperature $T = 1/\beta$ one has [3]

$$\begin{aligned} \langle a_k^\dagger a_{k'} \rangle_\beta &= (2\pi)^3 2\omega_k N(\omega_k)\delta(\vec{k}-\vec{k}'), \\ \langle a_k a_{k'}^\dagger \rangle_\beta &= (2\pi)^3 2\omega_k [1+N(\omega_k)]\delta(\vec{k}-\vec{k}'), \end{aligned} \tag{2}$$

where $N(k_0) = [\exp(\beta k_0) - 1]^{-1}$ is the Bose-Einstein distribution function. The thermal averages in eqs. (2) are both temperature-dependent. In particular, in the second line, we have a "thermal" contribution, the $N(\omega_k)$ inside the square brackets, and a $T = 0$, "quantum" contribution, the "1". Of course, the commutator $[a_k, a_{k'}^\dagger]$ being a c-number, its expectation value is independent on the state and is the same as in the zero-temperature vacuum

$$\langle [a_k, a_{k'}^\dagger] \rangle_\beta = (2\pi)^3 2\omega_k\, \delta(\vec{k}-\vec{k}'), \tag{3}$$

independent of T. We propose to modify the above relations by introducing an infrared cutoff *on the thermal part only*, that is

$$\begin{aligned} \langle a_k^\dagger a_{k'} \rangle_\beta^\Lambda &= (2\pi)^3 2\omega_k N(\omega_k)\Theta(|\vec{k}|,\Lambda)\delta(\vec{k}-\vec{k}'), \\ \langle a_k a_{k'}^\dagger \rangle_\beta^\Lambda &= (2\pi)^3 2\omega_k [1+N(\omega_k)\Theta(|\vec{k}|,\Lambda)]\delta(\vec{k}-\vec{k}'), \end{aligned} \tag{4}$$

where, in general, $\Theta(|\vec{k}|,\Lambda)$ is a cutoff function which is 1 for $|\vec{k}| \geq \Lambda$ and rapidly vanishes for $|\vec{k}| < \Lambda$. The simplest choice for $\Theta(|\vec{k}|,\Lambda)$ is the Heavyside theta function

$$\Theta(|\vec{k}|,\Lambda) \to \theta(|\vec{k}| - \Lambda)\,. \tag{5}$$

For simplicity, in the following we will always use the step function, even though in order to perform safe formal manipulations one should choose [10] a cutoff function which is always non-vanishing and of class C^∞, and take only at the end the limit (5).

The physical state described by the relations in (4) is the one in which thermal equilibrium is achieved only by the fast, high frequency, modes, from $|\vec{k}| \to \infty$ down to the cutoff scale $|\vec{k}| \simeq \Lambda$. The corresponding density matrix is given by

$$\rho = c \exp\left[-\int d^3k\ \beta_{k,\Lambda}\, a_k^\dagger a_k \right], \tag{6}$$

where $\beta_{k,\Lambda} = \beta\omega_k$ for $|\vec{k}| > \Lambda$ and goes to infinity for $|\vec{k}| < \Lambda$. The quantum relation

$$\langle [a_k, a_{k'}^\dagger] \rangle_\beta^\Lambda = (2\pi)^3 2\omega_k\, \delta(\vec{k}-\vec{k}'), \tag{7}$$

analogous to eq. (3), is valid, in this state, for any value of the momenta, irrespective of the cutoff. Note that this would have not been the case if we had multiplied the whole right-hand side of the second eq. (2) for the cutoff function $\Theta(|\vec{k}|, \Lambda)$.

Now the cutoff Green functions are defined in the usual way, as statistical averages of Heisenberg fields ordered along a path C in the complex time plane [5, 6], running from $-\infty$ to $+\infty$ along the real time axis and back from $+\infty - i\varepsilon$ to $-\infty - i\varepsilon$ infinitesimally below it. As we will show in Appendix A, the effect of the non-thermal density matrix (6) can be accounted for by simply working in the usual equilibrium formalism, but with the Bose-Einstein distribution function multiplied by the cutoff function. For instance, the two-point Green function is

$$G^{(c)}(x, x') = \theta_c(t - t')C^{>}(x, x') + \theta_c(t' - t)C^{<}(x, x'), \tag{8}$$

where $\theta_c(t-t')$ is the generalization of the theta function on the contour C and the two-point correlation functions are

$$C^{>}(x, x') = \langle \hat{\phi}(x)\hat{\phi}(x')\rangle_{\beta}^{\Lambda} = C^{<}(x', x)\,. \tag{9}$$

The free cutoff propagator $D_{\Lambda}^{(c)}(x - x') = -iG_0^{(c)}(x, x')$ can be computed in the standard way, by substituting (1) in (8) and (9), and using (4). One obtains

$$\begin{aligned} iD_{\Lambda}^{(c)}(x - x') = &\int \frac{d^4k}{(2\pi)^4} 2\pi\delta(k^2 - m^2)e^{-ik\cdot(x-x')} \\ &\times \left[\theta(k_0)\theta_c(t - t') + \theta(-k_0)\theta_c(t' - t) + N(|k_0|, \Lambda)\right], \end{aligned} \tag{10}$$

where

$$N(|k_0|, \Lambda) = N(|k_0|)\ \theta(|\vec{k}| - \Lambda)\,. \tag{11}$$

The propagator can be seen as a 2×2 matrix whose components are given by

$$\begin{array}{ll} D_{\Lambda}^{(11)}(t - t') = & D_{\Lambda}^{(c)}(t - t'), \\ D_{\Lambda}^{(22)}(t - t') = & D_{\Lambda}^{(c)}((t - i\varepsilon) - (t' - i\varepsilon)), \\ D_{\Lambda}^{(12)}(t - t') = & D_{\Lambda}^{(c)}(t - (t - i\varepsilon)), \\ D_{\Lambda}^{(21)}(t - t') = & D_{\Lambda}^{(c)}((t - i\varepsilon) - t'), \end{array} \tag{12}$$

where t, t' lie on the real time axis and we have omitted the spatial arguments of the propagator.

Using expression (10) and then Fourier transforming, we obtain the RT cutoff propagator in momentum space

$$D_{\Lambda}(k) = \begin{pmatrix} \Delta_0 & (\Delta_0 - \Delta_0^*)\theta(-k_0) \\ (\Delta_0 - \Delta_0^*)\theta(k_0) & -\Delta_0^* \end{pmatrix} + (\Delta_0 - \Delta_0^*)\, N(|k_0|, \Lambda)\, B, \tag{13}$$

where

$$\Delta_0 = \frac{1}{k^2 - m^2 + i\varepsilon} \tag{14}$$

and

$$B = \begin{pmatrix} 1 & 1 \\ & \\ 1 & 1 \end{pmatrix} . \tag{15}$$

Note the separation between the $T = 0$ and the thermal part. In the $\varepsilon \to 0$ limit, we have

$$\Delta_0 - \Delta_0^* \longrightarrow -2i\pi\delta(k^2 - m^2), \tag{16}$$

that is, the thermal part of the tree level propagator involves on-shell degrees of freedom only (only real particles belong to the thermal bath).

Since it will be useful in the following, we give here also the expression for the derivative of the propagator with respect to the cutoff

$$\frac{\partial}{\partial\Lambda} D_\Lambda(k) = 2i\pi\delta(k^2 - m^2)\,\delta(|\vec{k}| - \Lambda)\,N(|k_0|)\;B \tag{17}$$

and for the inverse of the propagator

$$\begin{aligned} D_\Lambda(k)^{-1} &= \frac{1}{\Delta_0\Delta_0^*} \begin{pmatrix} \Delta_0^* & (\Delta_0 - \Delta_0^*)\theta(-k_0) \\ & \\ (\Delta_0 - \Delta_0^*)\theta(k_0) & -\Delta_0 \end{pmatrix} \\ &- \frac{\Delta_0 - \Delta_0^*}{\Delta_0\Delta_0^*} N(|k_0|, \Lambda) \begin{pmatrix} 1 & -1 \\ & \\ -1 & 1 \end{pmatrix} . \end{aligned} \tag{18}$$

2.2 Cutoff effective action

By introducing independent sources j_1 and j_2 for the two pieces of the contour C (the real time axis and the $t - i\varepsilon$ one, respectively), the path integral representation of the generating functional of RT cutoff Green functions can be written as [7]

$$Z_\Lambda[j] = \int [d\phi_1][d\phi_2] \exp i \left\{ \tfrac{1}{2} \text{tr}\ \phi \cdot D_\Lambda^{-1} \cdot \phi + S[\phi] + \text{tr}\ j \cdot \phi \right\}, \tag{19}$$

where the trace means integration over momenta while the dot represents the sum over field indices

$$\begin{aligned} \text{tr}\ \phi \cdot D_\Lambda^{-1} \cdot \phi &= \sum_{a,b=1,2} \int \frac{d^4p}{(2\pi)^4} \phi_a(p)\,[D^{-1}(p)]_{ab}\phi_b(-p), \\ \text{tr}\ j \cdot \phi &= \sum_{a=1,2} \int \frac{d^4p}{(2\pi)^4}\, j_a(-p)\phi_a(p) \end{aligned} \tag{20}$$

and $S[\phi]$ is the bare interaction action

$$S[\phi] = S[\phi_1] - S[\phi_2]. \tag{21}$$

Notice that in eq. (19) it is assumed that the usual procedure of renormalization of the perturbative ultraviolet divergences of the zero-temperature theory has been carried out.

Namely we assumed a regulator, whose precise form is irrelevant for our discussion, and a set of zero-temperature renormalization conditions.

In the usual interpretation of the RT formalism, ϕ_1 and ϕ_2 are respectively the "physical" field and the "ghost" field.

From the expression of the propagator in (13) one can immediately realize that, in the limit $\Lambda \to \infty$ (which in practice, due to the exponential behaviour of the Bose-Einstein distribution function, means $\Lambda \gg T$), none of the thermal modes are propagating inside the loops. In this case by taking the functional derivatives of (19) with respect to the source j_1 one obtains exactly the zero-temperature, fully renormalized, quantum field theory. On the other hand, in the opposite limit $\Lambda \to 0$ the propagator tends to the equilibrium, Real Time, finite temperature propagator, and consequently the generating functional in (19) gives the full finite temperature theory in thermal equilibrium.

We define as usual the cutoff effective action as the Legendre transform of the generating functional of the connected Green functions $W_\Lambda[j] = -i \log Z_\Lambda[j]$ [11, 12, 13]:

$$\tfrac{1}{2} \text{tr } \phi \cdot D_\Lambda^{-1} \cdot \phi + \Gamma_\Lambda[\phi] = W_\Lambda[j] - \text{tr } j \cdot \phi, \qquad \phi = \frac{\delta W_\Lambda[j]}{\delta j}\,, \tag{22}$$

where we have isolated the free part of the cutoff effective action and used for the classical fields the same notation as for the quantum fields.

As discussed in [2], the free energy of the system is given by the functional $\bar{\Gamma}_\Lambda[\varphi]$, defined by the relation

$$\frac{\delta \bar{\Gamma}_\Lambda[\varphi]}{\delta \varphi} = \left.\frac{\delta \Gamma_\Lambda[\phi]}{\delta \phi_1}\right|_{\phi_1=\phi_2=\varphi}. \tag{23}$$

The tadpole $\bar{\Gamma}^{(1)}_\Lambda(\varphi)$, which will play an important rôle in the following, is found by evaluating (23) for a space-time independent field configuration

$$\bar{\Gamma}^{(1)}_\Lambda(\varphi) = \left.\frac{\delta \bar{\Gamma}_\Lambda[\varphi]}{\delta \varphi}\right|_{\varphi=\text{const.}}. \tag{24}$$

We conclude this subsection by recalling an important property. By inspection, one finds that the cutoff effective action has a discrete Z_2 symmetry [2]

$$\Gamma_\Lambda[\phi_1, \phi_2] = -\Gamma^*_\Lambda[\phi^*_2, \phi^*_1]\,. \tag{25}$$

This relation has the following consequences on the derivatives of the tadpole

$$\frac{\partial \bar{\Gamma}^{(1)}_\Lambda(\varphi)}{\partial \varphi} = \left.\frac{\delta^2 \Gamma_\Lambda[\phi]}{\delta \phi_1^2}\right|_{\phi_1=\phi_2=\varphi}, \tag{26}$$

$$\frac{\partial^2 \bar{\Gamma}^{(1)}_\Lambda(\varphi)}{\partial \varphi^2} = \sum_{a,b=1,2} \left.\frac{\delta^3 \Gamma_\Lambda[\phi]}{\delta \phi_1 \delta \phi_a \delta \phi_b}\right|_{\phi_1=\phi_2=\varphi}, \tag{27}$$

where again φ is space-time independent. We will use these relations below.

2.3 Relation to the other formulations of the Wilson RG

Before proceeding with the discussion, we would like to comment on the relationship between our approach and the other formulations of the Wilson RG existing in the literature.

The first application of the continuum Wilson RG philosophy to quantum field theory at $T = 0$ is due to Polchinski [10]. The motivation of that work was to provide a demonstration of perturbative renormalizability without resorting to diagrammatic techniques.

In this formulation, which we will indicate in the following as the Polchinski RG, only the modes between an infra-red cutoff Λ and an ultraviolet cutoff Λ_0 are allowed to propagate inside loops. The bare Euclidean Lagrangian depends only on the UV scale Λ_0, which is eventually sent to infinity, and the renormalization group flow describes the effect of quantum fluctuations coming into play as the infra-red cutoff Λ is lowered. This can be achieved in a way formally analogous to what we have done in this section, the main difference being in the form of the cutoff propagator. Namely, in this case the cutoff procedure has to be applied to the full propagator, and not only to a part of it, as we have done here. That is

$$D_{\Lambda\Lambda_0}(k) = \Theta(k, \Lambda)\Theta(\Lambda_0, k)D(k) \ , \tag{28}$$

where $D(k)$ is the tree level propagator. The internal legs of the Green functions generated by the analogues of our Z_Λ, W_Λ, and Γ_Λ, now carry only momenta $\Lambda < k < \Lambda_0$. Therefore the 1PI Green functions are given by the bare couplings for $\Lambda = \Lambda_0$ (since no modes are propagating inside loops) and by the fully renormalized ones for $\Lambda = 0$ (since in this limit the quantum corrections have been included at any momentum scale).

On the other hand, since the cutoff procedure described in this section modifies the thermal part of the propagator only, the Green functions defined in this paper contain the full quantum corrections, independently of the value of Λ. Lowering Λ, we are introducing new *thermal* modes only.

In short, the Polchinski formulation of the Wilson RG interpolates between the *bare* and the *physical* (renormalized) theory, whereas the formulation presented in this paper interpolates between the *physical* theory at $T = 0$ and the *physical* theory at $T \neq 0$.

Since the two RG's describe two different physical problems, also the boundary conditions will be different. As stated previously, at $\Lambda = \Lambda_0$ in the Polchinski RG we have the bare Lagrangian. Thus, one should impose two different classes of boundary conditions [10, 12]. The first one involves all the "irrelevant" operators, the ones with dimension larger than 4. These should satisfy the power counting in Λ_0 at the ultraviolet scale $\Lambda = \Lambda_0$. The remaining, "relevant", couplings have to be fixed by the matching of the renormalized theory and the physical observables, and so the initial conditions for them should be given at $\Lambda = 0$.

On the other hand, in the approach presented in this paper, the physical theory corresponds to the initial condition at $\Lambda \to \infty$, so that all the couplings, both the relevant and the irrelevant ones, have to be fixed at this point.

The Wilson RG method has already been applied to the study of the $T \neq 0$ quantum field theory in the IT formalism in ref. [16] and more recently in [17]. The thermal

propagator in the IT formalism [18] is obtained from the Euclidean propagator after the replacement $k_0 \to \omega_n$, where $\omega_n = 2\pi T n$ are the Matsubara frequencies

$$\frac{1}{k^2 + m^2} \to \frac{1}{(2\pi T n)^2 + |\vec{k}|^2 + m^2} \tag{29}$$

and the k_0 integration is consistently replaced by $T \sum_n$.

While in the RT formalism the quantum and the thermal part of the propagator can be clearly identified, as in (13), in the Imaginary Time formalism this separation cannot be achieved, as we read from (29). So, in order to formulate a Wilson RG in this case, we can only modify the thermal propagator as in (28), imposing an IR cutoff on the combination $(2\pi T n)^2 + |\vec{k}|^2$.

In this case we recover the bare theory for $\Lambda = \Lambda_0$, since no modes, either quantum or thermal, have been integrated in the loops, while for $\Lambda = 0$ we have the physical theory at $T \neq 0$. This formulation of the RG describes the effect of thermal and quantum fluctuations at the same time, and it is different from both Polchinski's and ours. Concerning the boundary conditions in this case, it should be noticed that since the RG now interpolates between the bare theory and the $T \neq 0$ physical theory, there is no value of Λ that corresponds to the physical theory at $T = 0$, the one that is supposed to be directly related to the measurable observables. Then, in order to match the running parameters with the physical theory, a preliminary step has to be performed, consisting in the derivation of the bare parameters from the renormalized, $T = 0$ theory [16, 17].

3 Renormalization group flow equations

In this section we study the Λ-dependence of the cutoff effective action. By taking the derivative with respect to Λ of (19) we obtain the evolution equation for $Z_\Lambda[j]$

$$\Lambda \frac{\partial}{\partial \Lambda} Z_\Lambda[j] = -\frac{i}{2}\,\mathrm{tr}\,\frac{\delta}{\delta j} \cdot \Lambda \frac{\partial}{\partial \Lambda} D_\Lambda^{-1} \cdot \frac{\delta}{\delta j} Z_\Lambda[j]\,. \tag{30}$$

Choosing as initial conditions for $Z_\Lambda[j]$ at $\Lambda \gg T$ the full renormalized theory at zero temperature, the above evolution equation describes the effect of the inclusion of the thermal fluctuations at the momentum scale $|\vec{k}| = \Lambda$.

The evolution equation for the generating functional of the connected Green functions $W_\Lambda[j] = -i \log Z_\Lambda[j]$ can be derived straightforwardly from eq. (30)

$$\Lambda \frac{\partial}{\partial \Lambda} W_\Lambda[j] = -\frac{i}{2}\mathrm{tr}\left[\Lambda \frac{\partial}{\partial \Lambda} D_\Lambda^{-1} \cdot \frac{\delta^2 W_\Lambda[j]}{\delta j\, \delta j}\right] + \frac{1}{2}\mathrm{tr}\,\frac{\delta W_\Lambda[j]}{\delta j} \cdot \Lambda \frac{\partial}{\partial \Lambda} D_\Lambda^{-1} \cdot \frac{\delta W_\Lambda[j]}{\delta j}\,. \tag{31}$$

By using this equation in (22), the flow equation for the cutoff effective action $\Gamma_\Lambda[\phi]$ is found:

$$\Lambda \frac{\partial}{\partial \Lambda} \Gamma_\Lambda[\phi] = \frac{i}{2}\mathrm{tr}\left[\Lambda \frac{\partial}{\partial \Lambda} D_\Lambda^{-1} \cdot \left(D_\Lambda^{-1} + \frac{\delta^2 \Gamma_\Lambda[\phi]}{\delta\phi\, \delta\phi}\right)^{-1}\right]. \tag{32}$$

Notice that the terms coming from the Λ-dependence of the classical field ϕ cancel out, which is a well-known general property of the Legendre transform. The evolution equations for the various vertices can be found by expanding the r.h.s. in powers of ϕ [12, 13].

Deriving eq. (32) with respect to ϕ_1 and then setting ϕ_1 and ϕ_2 equal to a constant background φ, we obtain the evolution equation for the tadpole, defined in eq. (24),

$$\Lambda\frac{\partial}{\partial\Lambda}\bar{\Gamma}^{(1)}_\Lambda(\varphi) = -\frac{i}{2}\mathrm{tr}\left\{\left[D_\Lambda^{-1}+\Sigma_\Lambda(\varphi)\right]^{-1}\cdot\Lambda\,\frac{\partial}{\partial\Lambda}D_\Lambda^{-1}\cdot\left[D_\Lambda^{-1}+\Sigma_\Lambda(\varphi)\right]^{-1}\cdot\Gamma^{(3)}_\Lambda(\varphi)\right\}, \quad (33)$$

where

$$\left[\Gamma^{(3)}_\Lambda(\varphi)\right]_{ij} = \left.\frac{\delta\Gamma_\Lambda[\phi]}{\delta\phi_i\,\delta\phi_j\,\delta\phi_1}\right|_{\phi_1=\phi_2=\varphi} \quad (34)$$

and we introduced the field-dependent self-energy matrix

$$(2\pi)^4\delta^{(4)}(p+p')\left[\Sigma_\Lambda(p;\,\varphi)\right]_{ij} = \left.\frac{\delta^2\,\Gamma_\Lambda[\phi]}{\delta\phi_i(p)\,\delta\phi_j(p')}\right|_{\phi_1=\phi_2=\varphi}. \quad (35)$$

First of all, we need to study the kernel of the evolution equation for the tadpole, that is

$$K_\Lambda(k;\,\varphi) \equiv -i\left[D_\Lambda^{-1}+\Sigma_\Lambda(\varphi)\right]^{-1}\cdot\Lambda\frac{\partial}{\partial\Lambda}D_\Lambda^{-1}\cdot\left[D_\Lambda^{-1}+\Sigma_\Lambda(\varphi)\right]^{-1}. \quad (36)$$

The kernel contains the "full" cutoff matrix propagator $\left[D_\Lambda^{-1}+\Sigma_\Lambda(\varphi)\right]^{-1}$. This can be obtained (by assuming a Schwinger-Dyson equation [7]) from eq. (13) by substituting Δ_0 with

$$\Delta_\Lambda = \Delta_0\sum_{n=0}^{\infty}(-\Pi_\Lambda\Delta_0)^n = \frac{1}{k^2-m^2+\Pi_\Lambda(k;\,\varphi)+i\varepsilon} \quad (37)$$

where the quantity $\Pi_\Lambda(k;\,\varphi)$ is related to the 11 component of the self-energy matrix by

$$\begin{cases} \mathrm{Re}\,\Pi_\Lambda(k;\,\varphi) &= \mathrm{Re}\,\left[\Sigma_\Lambda(k;\,\varphi)\right]_{11} \\ \\ \mathrm{Im}\,\Pi_\Lambda(k;\,\varphi) &= \dfrac{1}{1+2N(|k_0|,\Lambda)}\,\mathrm{Im}\,\left[\Sigma_\Lambda(k;\,\varphi)\right]_{11} \end{cases} \quad (38)$$

Collecting the above formulae we can compute the kernel, which turns out to be

$$K_\Lambda(k;\,\varphi) = -i(\Delta_0-\Delta_0^*)\frac{\Delta_\Lambda\Delta_\Lambda^*}{\Delta_0\Delta_0^*}\,\Lambda\,\delta(|\vec{k}|-\Lambda)\,N(|k_0|)\,B\,. \quad (39)$$

Notice that if one uses a cutoff function different from the step function, the kernel is obtained from (39) by substituting $\delta(|\vec{k}|-\Lambda)$ with the Λ-derivative of the new cutoff function.

Comparing (39) with the derivative with respect to Λ of the tree level propagator, which we have computed in (17), we see that the delta function which forces the momenta on the zero temperature mass-shell has been replaced by

$$i(\Delta_0-\Delta_0^*)\frac{\Delta_\Lambda\Delta_\Lambda^*}{\Delta_0\Delta_0^*} = -2\pi\rho_\Lambda(k;\varphi)+i\Delta_0\Delta_0^*(\Pi_\Lambda-\Pi_\Lambda^*)(1+R_\Lambda+R_\Lambda^*+R_\Lambda R_\Lambda^*), \quad (40)$$

where

$$\rho_\Lambda(k;\varphi) = -\frac{i}{2\pi}\left[\Delta_\Lambda - \Delta_\Lambda^*\right] \tag{41}$$

is the interacting spectral function and

$$R_\Lambda = \sum_{n=1}^{\infty} \left(\Pi_\Lambda \Delta_0\right)^n .$$

The second contribution to the kernel in eq. (40) exhibits pinch singularities of the form $\Delta_0^m\ \Delta_0^{*n}$. It is also proportional to the imaginary part of the self-energy $\Pi_\Lambda(k)$, which is non-vanishing along the whole real axis in the k_0 complex plane[1]. As will be shown in Appendix B, these pinch singularities cancel with analogous contributions coming from $\Gamma_\Lambda^{(3)}$ once the kernel (40) is inserted in the evolution equation for the tadpole to give

$$\Lambda\frac{\partial}{\partial\Lambda}\bar{\Gamma}_\Lambda^{(1)}(\varphi) = -i\frac{\Lambda}{2}\int \frac{d^4k}{(2\pi)^4}\,\delta(|\vec{k}|-\Lambda)(\Delta_0-\Delta_0^*)\frac{\Delta_\Lambda\Delta_\Lambda^*}{\Delta_0\Delta_0^*}\,N(|k_0|)\bar{\Gamma}_\Lambda^{(3)}(k;-k;\varphi)\ , \tag{42}$$

where

$$\bar{\Gamma}_\Lambda^{(3)}(k;k';\varphi) \equiv \frac{\partial^2\bar{\Gamma}_\Lambda^{(1)}(\varphi)}{\partial\varphi(k)\partial\varphi(k')} = \mathrm{Tr}\left\{B\cdot\Gamma_\Lambda^{(3)}(k;k';\varphi)\right\} \tag{43}$$

(here the trace is over $1,2$ field indices) and $\Gamma_\Lambda^{(3)}$ has been defined in (34).

It is important to stress here that the above equation is exact, since no approximation such as perturbative expansion or truncation has been performed up to now.

In the rest of this section we will discuss the evolution equation obtained neglecting the imaginary part of the self energy on-shell, which in perturbation theory arises only at two-loops [14, 15]. We will use this approximation as the starting point for the numerical analysis which we will describe in sect. 4. In this approximation, the kernel in (40) can be written as

$$\frac{i}{k^2 - m^2 + \mathrm{Re}\,\Pi_\Lambda(k;\,\varphi) + i\varepsilon} - \frac{i}{k^2 - m^2 + \mathrm{Re}\,\Pi_\Lambda(k;\,\varphi) - i\varepsilon}, \tag{44}$$

so that the only singularities are the poles in the k_0 complex plane satisfying $k_0^2 = |\vec{k}|^2 + m^2 - \mathrm{Re}\,\Pi_\Lambda(k;\,\varphi) \pm i\varepsilon = 0$, and it vanishes in the rest of the complex plane. Notice that the pinch singularities have disappeared in this approximation.

In the case in which $|\vec{k}|^2 + m^2 - \mathrm{Re}\,\Pi_\Lambda(k;\,\varphi) > 0$ the poles lie infinitesimally close to the real axis, and (44) reduces to $2\pi\delta(k^2 - m^2 + \mathrm{Re}\,\Pi_\Lambda(k;\,\varphi))$. On the other hand, if $|\vec{k}|^2 + m^2 - \mathrm{Re}\,\Pi_\Lambda(k;\,\varphi) < 0$, as can be the case when there is spontaneous symmetry breaking, the poles lie on the imaginary axis, and the contributions of the two pieces in (44) to the integration along the real k_0 axis cancel.

Then, the quantity (40) appearing in the kernel reduces, in the $\varepsilon \to 0$ limit, to

$$i(\Delta_0-\Delta_0^*)\frac{\Delta_\Lambda\Delta_\Lambda^*}{\Delta_0\Delta_0^*} \to 2\pi\delta(k^2 - m^2 + \mathrm{Re}\,\Pi_\Lambda(k;\,\varphi))\,\theta(|\vec{k}|^2 + m^2 - \mathrm{Re}\,\Pi_\Lambda(k;\,\varphi))\,. \tag{45}$$

The physical meaning of this kernel is straightforward. As the infrared cutoff Λ is lowered, the mass associated to the new modes coming into thermal equilibrium is not given by

[1] We thank S. Jeon for drawing our attention on this point.

the $T = 0$ pole mass m^2, but by the thermal mass, that is the pole of the full propagator obtained by integrating over the high momentum modes ($|\vec{k}| > \Lambda$), already in thermal equilibrium. This comes out quite naturally in this formalism, whereas in perturbation theory one has to perform ad hoc resummations in order to cure the infrared divergences. We will come back to the comparison between this approach and perturbation theory in the following.

The other interesting feature of this kernel has to do with the case of spontaneous symmetry breaking. In perturbation theory, one is not allowed to compute the effective potential for values of the background field close to the symmetric phase, if the temperature is less than the critical temperature of the phase transition. This is because the thermal mass squared is negative, in this region, which gives rise to a complex effective potential. As showed by Weinberg and Wu [19], the real part of this effective potential can still be interpreted as the energy density of a spatially homogeneous, although unstable, state.

On the other hand, in this approach, we see that the only modes giving rise to a thermal evolution are those with a real energy, and the running stops as soon as the energy squared becomes negative, so that no imaginary parts for the effective potential are generated.

Using (45), the evolution equation for the tadpole (42) now takes the simple form,

$$\Lambda\frac{\partial}{\partial\Lambda}\bar{\Gamma}^{(1)}_{\Lambda}(\varphi) = -\pi\Lambda\int\frac{d^4k}{(2\pi)^4}\,\delta(|\vec{k}|-\Lambda)\delta(k^2-m^2+\mathrm{Re}\,\Pi_\Lambda(k;\,\varphi'))\,N(|k_0|) \\ \times\theta(|\vec{k}|^2+m^2-\mathrm{Re}\,\Pi_\Lambda(k;\,\varphi))\,\bar{\Gamma}^{(3)}_{\Lambda}(k;-k;\varphi)\;, \tag{46}$$

which will be studied numerically in the next section in an approximation scheme based on a derivative expansion and truncations.

Before concluding this section, we would like to briefly discuss the infra-red limit of the evolution equation for the tadpole (46). As the energy of the thermal modes becomes much smaller than T, the Bose-Einstein distribution function can be approximated as

$$N(|k_0|) \simeq \frac{T}{|k_0|}\left(1-\frac{1}{2}\frac{|k_0|}{T}+\ldots\right). \tag{47}$$

Integrating in k_0 in (42) and keeping only the first term in the expansion for $N(|k_0|)$ in (47) the evolution equation for the tadpole becomes

$$\Lambda\frac{\partial}{\partial\Lambda}\bar{\Gamma}^{(1)}_{\Lambda}(\varphi) \simeq -\frac{\Lambda\,T}{2}\int\frac{d^3k}{(2\pi)^3}\,\delta(|\vec{k}|-\Lambda)\frac{\bar{\Gamma}^{(3)}_{\Lambda}(\vec{k};-\vec{k};\varphi)}{|\vec{k}|^2+m^2+\mathrm{Re}\,\Pi_\Lambda(\vec{k};\,\varphi)} \tag{48}$$

if

$$\omega^2_\Lambda = \Lambda^2+\mathrm{Re}\,\Pi_\Lambda(k_0=\omega_\Lambda,|\vec{k}|=\Lambda;\,\varphi) \ll T^2. \tag{49}$$

Equation (48) shows that, in the limit of eq. (49), and neglecting the imaginary part of the self-energy, the exact four-dimensional running at finite temperature reduces to a purely three-dimensional one, at zero temperature. The momentum integral is now a genuine three-dimensional loop integral with the zero-temperature propagator. Equation (48) is, in fact, the same equation as would have been obtained in the Polchinski RG in three

dimensions and $T = 0$, provided the following matching between the three-dimensional and the four-dimensional couplings had been done:

$$\begin{aligned}
\bar{\Gamma}^{(1)}_{\Lambda}(\varphi)_{D=3} &= \frac{1}{\sqrt{T}}\,\bar{\Gamma}^{(1)}_{\Lambda}(\varphi)_{D=4} \\
\mathrm{Re}\,\Pi_{\Lambda}(\vec{k};\,\varphi)_{D=3} &= \mathrm{Re}\,\Pi_{\Lambda}(k_0=\omega_k,\vec{k};\,\varphi)_{D=4} \\
\bar{\Gamma}^{(3)}_{\Lambda}(\vec{k};\,-\vec{k};\varphi)_{D=3} &= \sqrt{T}\,\bar{\Gamma}^{(3)}_{\Lambda}(k_0=\omega_k,\vec{k};\;k_0'=\omega_k,-\vec{k};\;\varphi)_{D=4}\,.
\end{aligned} \tag{50}$$

Obviously, the above matching has to be performed at a value of Λ such that the condition (49) is fulfilled, that is in the three-dimensional regime. So, the question arises whether we can neglect, in a first approximation, the four-dimensional running from $\Lambda = \Lambda_0 \gg T$ to some value of Λ, Λ_{3D}, which is inside the three-dimensional regime. The answer is clearly negative, as we can read from Fig. 1.

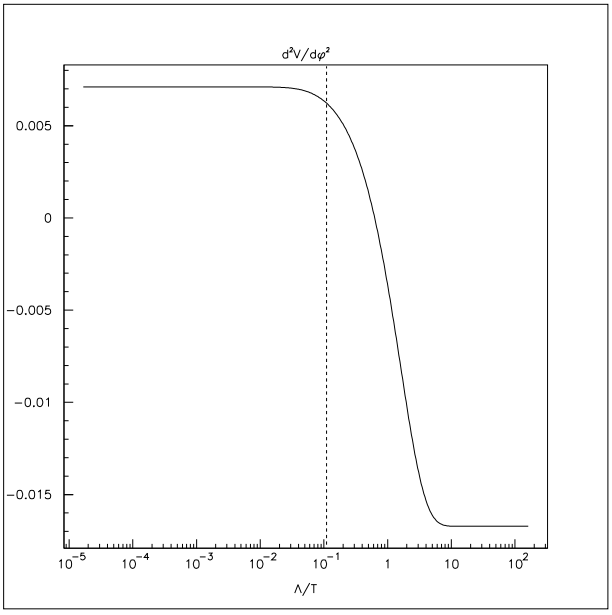

Figure 1: The running of the thermal mass ($V''_{\Lambda}(\varphi = 0)$, here given in units of v^2, v being the zero temperature vev) with respect to the ratio Λ/T. The region to the left of the dashed line corresponds to the three-dimensional regime (see text). We have fixed $T = 1.2\,T_c$ and $\lambda = 0.1$.

In this picture, we have plotted the value of the thermal mass in $\varphi = 0$ (see Sect. 5) as a function of the ratio Λ/T. The area to the left of the dashed line represents the region in which the Bose-Einstein distribution function can be approximated by $T/|k_0|$ with an accuracy better than 10%, and consequently the three-dimensional RG equation (48) is valid to that accuracy.

As we can see, most of the running of the mass, from the zero-temperature value (corresponding to $\Lambda/T \gg 1$) to the finite temperature one, takes place before the three-dimensional region is reached. In other words, in order to match the three-dimensional

couplings with the physical four-dimensional ones, it is essential to keep the pure four-dimensional running (at $T \neq 0$) under control, and the study of the pure three-dimensional running is, in general, not justified.

On the other hand, dimensional reduction is justified when one is interested in the study of the critical theory. By universality, we know that this theory is insensitive to the "initial conditions" of the RG equations, so that the matching with the physical theory at zero temperature is not very important for the computation of universal quantities.

In our numerical analysis we will always consider the four-dimensional RG equations derived in the previous sections, without dimensional reduction. This approach will allow us to study both the universal quantities (critical exponents) and the non-universal ones (critical temperature, thermal masses), and to keep the transition from the four-dimensional to the effectively three-dimensional critical theory under control. At the end of sect. 5 we will comment on the possibility of using a dimensionally-reduced form of eq. (42) in order to study critical regime of the theory.

4 Derivative expansion, truncations, and comparison with perturbation theory

An analytic solution of the exact evolution equation for the tadpole in eq. (42) is not available. In principle one has to know the momentum dependence of both the full self-energy and the vertex $\bar{\Gamma}^{(3)}_\Lambda$, i.e. of the first and second derivative of the tadpole with respect to the field φ. A similar problem is encountered in the applications of the Polchinski RG in the zero-temperature field theory. An approximation scheme based on a derivative expansion (or momentum-scale expansion [20]) has proved to be very efficient in that context, at least in the case of the scalar theory.

In order to perform a systematic derivative expansion in this case, we expand the free energy functional defined in (23) in derivatives of the field $\varphi(x)$ as follows:

$$\bar{\Gamma}_\Lambda[\varphi] = \int d^4x \left[\tfrac{1}{2}m^2\varphi^2 - V_\Lambda(\varphi) + \tfrac{1}{2}(\partial\varphi)^2 Z_\Lambda(\varphi) + \tfrac{1}{2}(u\cdot\partial\varphi)^2 Y_\Lambda(\varphi) + \cdots\right] \qquad (51)$$

where the dots indicate higher derivative terms. Note the term containing the quadrivector of the thermal bath, u_μ, which has to be introduced for relativistic covariance.

Stopping the expansion in (51) at some order, plugging it into eq. (42), and equating the coefficients of the terms with the same powers of derivatives of the field, we obtain evolution equations for the functions V_Λ, Z_Λ, Y_Λ, and higher orders.

A remark is in order. Usually in the Polchinski RG the expansion (51) is not well defined if one uses a sharp cutoff function. The problem arises because in the zero external momentum limit the analogue of our eq. (32) and its momentum derivatives contain ill-defined products of $\delta(|\vec{k}| - \Lambda)$, coming from the derivative of the cut-off propagator, and $\theta(|\vec{k}| - \Lambda)$, coming from the underived propagator [20]. Fortunately, this is not the case in our approach. The point is, again, that the theta function here appears only in the thermal

part of the propagator, and all the above mentioned products of theta and delta functions cancel one another by the same mechanism that ensures the cancellation of the "pinch" singularities in the RT formalism. As a consequence, the sharp cutoff limit can safely be taken even at higher orders in the derivative expansion.

At the lowest order, the derivative expansion corresponds to neglecting the momentum dependence of the self-energy and the three-point vertex appearing in the r.h.s. of (42). In this approximation we can use (26) in (37) and Δ_Λ can be written as $\Delta_\Lambda = [k^2 - V''_\Lambda(\varphi)]^{-1}$, where the prime over the effective potential indicates derivation with respect to the field. Analogously the three-point vertex in (43) is equal to $-V'''_\Lambda(\varphi)$. Note that since the imaginary part of the self-energy vanishes for zero external momenta, it does not contribute to the zero order in the derivative expansion. We get

$$\Lambda\frac{\partial}{\partial\Lambda}V'_\Lambda(\varphi) = -\frac{\Lambda^3}{4\pi^2}\,\frac{N(\omega_\Lambda(\varphi))}{\omega_\Lambda(\varphi)}\,V'''_\Lambda(\varphi)\,\theta(\Lambda^2+V''_\Lambda(\varphi))\,, \tag{52}$$

where

$$\omega_\Lambda(\varphi) = \sqrt{\Lambda^2 + V''_\Lambda(\varphi)}\,.$$

Using the equilibrium Bose-Einstein distribution for N and integrating in φ we find the following evolution equation for the effective potential

$$\Lambda\frac{\partial}{\partial\Lambda}V_\Lambda(\varphi) = -T\frac{\Lambda^3}{2\pi^2}\log\left[1-\exp\left(-\beta\sqrt{\Lambda^2+V''_\Lambda(\varphi)}\right)\right]\,\theta(\Lambda^2+V''_\Lambda(\varphi)). \tag{53}$$

The same equation was found in the Matsubara formalism in [21].

Looking at the effective potential as a function of the two variables Λ and φ we see that the evolution equation is a non-linear partial derivative differential equation, with the initial condition $V_{\Lambda=\infty}(\varphi)$ being the renormalized effective potential at zero temperature.

In principle, one could seek for a numerical solution of the evolution equation (52). An alternative procedure, which will be followed in the present paper, is to take further derivatives of eq. (52) with respect to the field φ, so that the partial derivative differential equation is turned into a infinite system of ordinary first order differential equations in Λ for the unknowns V'_Λ, V''_Λ, V'''_Λ, ..., with φ-dependent coefficients,

$$\left\{\begin{array}{rcl}
\Lambda\dfrac{\partial}{\partial\Lambda}V'_\Lambda(\varphi) &=& -\dfrac{\Lambda^3}{4\pi^2}\,\dfrac{N(\omega_\Lambda)}{\omega_\Lambda}\,V'''_\Lambda(\varphi)\,\theta(\Lambda^2+V''_\Lambda(\varphi)) \\[2ex]
\Lambda\dfrac{\partial}{\partial\Lambda}V''_\Lambda(\varphi) &=& -\dfrac{\Lambda^3}{4\pi^2}\,\left\{\dfrac{N(\omega_\Lambda)}{\omega_\Lambda}\,V''''_\Lambda(\varphi) + \dfrac{d}{dV''_\Lambda}\left[\dfrac{N(\omega_\Lambda)}{\omega_\Lambda}\right]\,[V'''_\Lambda(\varphi)]^2\right\}\,\theta(\Lambda^2+V''_\Lambda(\varphi)) \\[2ex]
&\equiv& -\Lambda\mathcal{F}_\Lambda\,[V''_\Lambda(\varphi)] \\
&\vdots&
\end{array}\right. \tag{54}$$

In the following we will solve this system of equations at different orders of truncations, in order to test the reliability of this further approximation (besides the derivative expansion).

This also allows us to make a comparison between the present approach and the perturbative one. The 1-loop perturbative result [22] and the daisy and super-daisy [8, 9, 23] resummed ones correspond to different approximations of the above system, truncated at most to second order, i.e. without taking into account the evolution of the third and higher order couplings. More precisely:

(i) 1-loop perturbation theory corresponds to a truncation to the first equation only, the one for the tadpole, in which V_Λ'' and V_Λ''' are approximated by their tree level values:

$$V_\Lambda''(\varphi) \simeq m^2(\varphi), \quad V_\Lambda'''(\varphi) \simeq \lambda\varphi \ ; \tag{55}$$

(ii) Daisy resummation improves the 1-loop result by partially taking into account the evolution of the "mass" V_Λ'', replacing $m^2(\varphi)$ with a Λ-independent "thermal mass" in the evolution equation for the tadpole. The trilinear and quadrilinear couplings are approximated by the tree level ones, that is

$$\begin{aligned} V_\Lambda''(\varphi) &\simeq m_T^2(\varphi) = m^2(\varphi) + \int_0^\infty d\Lambda' \mathcal{F}_{\Lambda'}\left[m^2(\varphi)\right] \\ V_\Lambda'''(\varphi) &\simeq \lambda\varphi \qquad\qquad V_\Lambda''''(\varphi) \simeq \lambda \end{aligned} \tag{56}$$

where the function $\mathcal{F}_\Lambda$, defined in (54), is computed by approximating $V_\Lambda''(\varphi)$ with the tree level mass.

(iii) Super-daisy resummation corresponds to an improvement of the previous approximation consisting in using as the value of the thermal mass the solution $\bar{m}_T$ of the "gap equation"[2]

$$\begin{aligned} V_\Lambda''(\varphi) &\simeq \bar{m}_T^2(\varphi) = m^2(\varphi) + \int_0^\infty d\Lambda' \mathcal{F}_{\Lambda'}\left[\bar{m}_T^2(\varphi)\right] \\ V_\Lambda'''(\varphi) &\simeq \lambda\varphi \qquad\qquad V_\Lambda''''(\varphi) \simeq \lambda. \end{aligned} \tag{57}$$

As we see, the (resummed) perturbative results do not take into account the evolution equations for the trilinear and higher orders couplings. Moreover, the Λ dependence of the second derivative of the effective potential is not considered, while a Λ-independent thermal mass is introduced.

In the next section, we will see how these effects are in general very important, mainly at temperatures close to the critical one. In particular, the inclusion of the running up to the fourth derivative of the potential, will change the transition from a (weakly) first-order one to a second-order one.

5 Numerical results

We have solved the system in eq. (54) at different orders of truncation and for different values of φ. In this way we are able to reconstruct the shape of the tadpole, and then of the

[2]Different resummation schemes have been presented in the literature, which however agree with one another up to $O(\beta)$, where $\beta = \lambda T/m_T$ is the effective expansion parameter.

effective potential. Since we are mainly interested in the study of the phase transition, we will solve the evolution equations for φ close to the origin, and running from $\Lambda = \Lambda_0 \gg T$ (in practice $\Lambda_0 > 10T$ will be enough, see Fig. 1) down to $\Lambda = 0$. As initial condition in $\Lambda = \Lambda_0$ we use the 1-loop effective potential at $T = 0$,

$$V_{\Lambda_0}(\varphi) = -\frac{1}{2}\mu^2\varphi^2 + \frac{\lambda}{4!}\varphi^4 + \frac{m^4(\varphi)}{64\pi^2}\left[\log\frac{m^2(\varphi)}{-2\mu^2} - \frac{3}{2}\right] , \tag{58}$$

where

$$m^2(\varphi) = -\mu^2 + \frac{\lambda}{2}\varphi^2 ,$$

and the potential has been renormalized in the $\overline{MS}$ scheme with $Q^2 = -2\mu^2$. In principle, different approximation schemes to the $T = 0$ effective potential could also be used, as given for instance by the Polchinski RG or by lattice computations. However, due to universality, the results for the critical theory should be almost insensitive to the details of the initial conditions, and the 1-loop approximation will be enough for our present purposes.

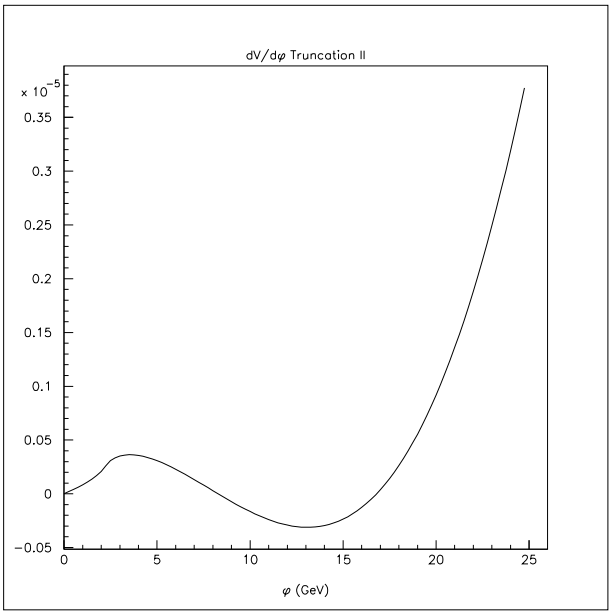

Figure 2: The tadpole $V'_{\Lambda=0}$ (in units of v^3) as a function of φ, obtained by truncating to order II. The temperature has been chosen such that the effective potential has two degenerate minima. We have fixed $v = 246$ GeV and $\lambda = 0.1$.

In Fig. 2 we show the tadpole $V'_{\Lambda=0}$ as a function of φ at the critical temperature, when the system has been truncated to the second line. As we see the potential has three stationary points (the zeros of $V'_{\Lambda=0}$), i.e. there are two minima, so that it describes a (weakly) first-order phase transition. In this case, we have fixed the temperature in such a way that the two minima are degenerate. This result agrees with the one obtained in 1-loop resummed perturbation theory, see for instance ref. [8].

However as soon as we turn the evolution for the third and fourth coupling on, as we did in Fig. 3, we see that things change dramatically. In this case the tadpole has

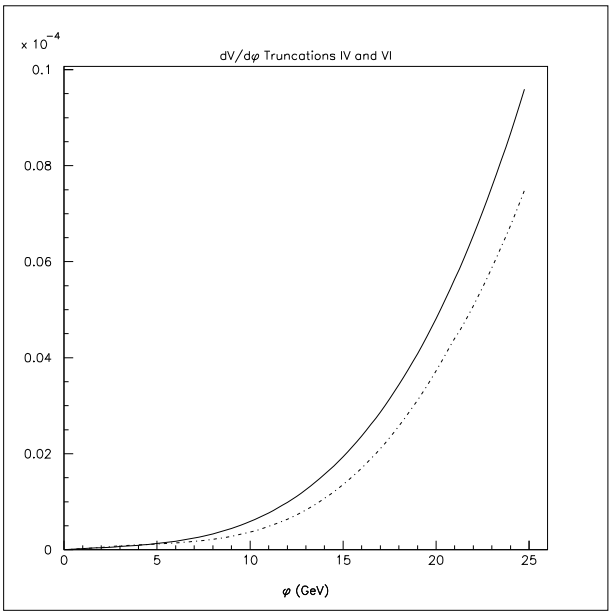

Figure 3: The tadpole $V'_{\Lambda=0}$ (in units of v^3) as a function of φ, obtained by truncating to order IV (continuous line) and VI (dash-dotted line). The temperature corresponds to the vanishing of the second derivative at the origin. We have fixed $v = 246$ GeV and $\lambda = 0.1$.

only one zero in $\varphi = 0$ for temperatures higher than a critical value and two zeros for lower temperatures. There is no temperature interval in which a third zero is present, in other words the effective potential develops no barrier between the symmetric and the asymmetric phases. The critical temperature T_c has been determined by requiring that the second derivative of the potential vanished. So, as soon as the running of the coupling is taken into account, the transition turns out to be of second-order, as would be expected from universality. The continuous and dash-dotted lines in this plot represent the results obtained by truncating to the fourth and sixth orders, respectively.

The running of the coupling is indeed a strong effect, in particular close to the critical temperature, as we read from Fig. 4. The $T = 0$ value for the coupling in this picture is chosen to be 0.1, and we see that, while for T far from T_c the running is an $\sim 10\%$ effect, for $T \to T_c$ the coupling vanishes.

The expansion parameter of the super-daisy resummation is given by $\beta = \lambda T/m_T$, where m_T is the mass scale of the theory, in this case the thermal mass $V''_{\Lambda=0}$. As $T \to T_c$ the thermal mass vanishes, and β diverges. This is the source of the wild infra-red problems of resummed perturbation theory, which prevent computations for temperatures close to the critical one. Including the running of the coupling, which is also vanishing at T_c, β tends to a finite value, and the infrared problems are, if not cured, at least domesticated. This observation was already made in ref. [16].

Looking at the evolution equations, we see that the range of Λ for which there is a sizeable running of the parameters is determined by the function $N(\omega_\Lambda)/\omega_\Lambda$. For $\Lambda \gg T$ the distribution function is exponentially suppressed and the running is negligible, as we

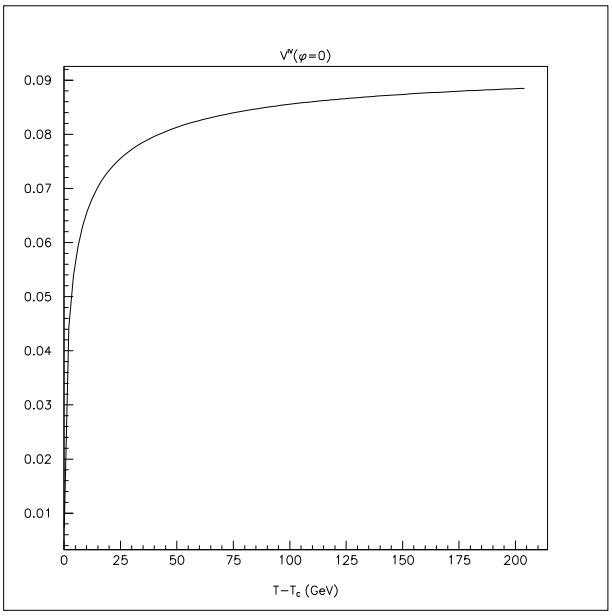

Figure 4: The running of the coupling constant $V''''_{\Lambda=0}(\varphi = 0)$ as a function of the temperature, for $T \geq T_c$. We have fixed $v = 246$ GeV and $\lambda = 0.1$.

read from Fig. 1. High energy modes are Boltzmann suppressed.

On the other hand, for small Λ, the function goes approximately as $T/(\Lambda^2 + V''_\Lambda)$, and so there is running as long as $\Lambda^2 > V''_\Lambda$. We can then define a correlation length as the inverse of Λ_c, such that

$$\Lambda_c^2 = V''_{\Lambda_c} \, . \tag{59}$$

Since there are no thermal modes at wavelengths larger than the correlation length, integrating over $\Lambda < \Lambda_c$ gives a negligible effect.

At the critical temperature we have $V''_{\Lambda=0} = 0$, and the correlation length diverges. We can define critical indices in the usual way (see for instance ref. [24]) and see how their computation is improved with respect to perturbation theory in the present approach.

At the lowest order in the derivative expansion, we have computed the critical exponent ν, governing the scaling of the renormalized mass near the critical temperature,

$$m_{\Lambda=0} \sim |T - T_c|^\nu \, , \tag{60}$$

and δ, describing how the magnetization at the critical temperature, $M(h, T = T_c) = \overline{\phi}$, scales with the magnetic field $h = V'_\Lambda$,

$$M(h, T = T_c) = \overline{\phi} \sim |h|^{1/\delta} = (V'_\Lambda)^{1/\delta}. \tag{61}$$

In Table 1 we summarize our results for the order of the phase transition, the critical temperature, and the critical exponents ν and δ obtained at the lowest order in the derivative expansion, and truncating the system in eq. (54) at second, fourth, and sixth order respectively.

Truncation	λ	T_c/v	Order trans.	ν	δ	η
II	0.1	1.99	I	–	–	–
IV	0.1	2.01	II	0.53	3.27	–
IV+Z_{II}	0.1	2.02	II	0.53	3.27	0.015
VI	0.1	2.02	II	0.58	3.57	–
Best values [24, 25]			II	0.63	4.82	0.032

Table 1: Results for the order of the phase transition, the critical temperature, and the critical exponents ν, δ, and η at different orders of approximations to the evolution equations (see text). The last row contains the best values in the literature.

In the last row we have listed the best results for the critical exponents obtained in the literature [24, 25]. We see that passing from the fourth to the sixth order of truncation we have a sizeable improvement in the result for ν and δ.

Including the corrections of $O\left((\partial\varphi)^2\right)$, keeping also the third and fourth terms in the expansion in (51), we can compute the critical exponent η, which describes the scaling of the two-point renormalized Green function with respect to Λ for $\Lambda \to 0$ at $T = T_c$:

$$\overline{\Gamma}^{(2)}_{\Lambda}(0) = m^2_{\Lambda}(1 + Z_{\Lambda}) \sim \Lambda^{2-\eta} \; . \tag{62}$$

Taking again derivatives with respect to the field φ, we obtain a system of first order differential equations for V'_{Λ}, V''_{Λ}, ..., Z_{Λ}, Z'_{Λ}, ..., Y_{Λ}, Y'_{Λ}, ... We have truncated this system at the fourth derivative for V_{Λ} and at the second one for Z_{Λ} and Y_{Λ}. Due to the Z_2 symmetry of the model, and using the fact that the initial condition for Y_{Λ} (which corresponds to its zero-temperature value) is zero, the effect of Y_{Λ} and its derivatives is negligible for values of φ close to the origin, and can be neglected with respect to that of Z_{Λ}. Moreover, we have still neglected the imaginary part of the self-energy on-shell, which is only a two-loop effect in standard perturbation theory [14, 15].

The result is reported in the fourth row of Table 1. As we can read, the effect of wave function renormalization is very small, as may be expected since in perturbation theory it arises at two loops. So, in this model, the derivative expansion turns out to be a very efficient approximation scheme.

The accuracy of our results for the critical exponents is generally a lot better than the one obtained in other approaches to the four-dimensional theory, such as perturbation theory or the IT formulation of the RG [16], both predicting $\nu = 0.5$.

Nevertheless, the best way to compute the critical exponents remains the study of the three-dimensional critical theory (see refs. [24, 25] and, for a RG computation, [26]). Once one has established that the effective potential solution of eq. (53) describes a second order phase transition, the universal quantities (the critical exponents) can be obtained studying the evolution around the fixed points of the critical ($T = T_c$) theory at large length-scales ($\Lambda \to 0$). For $\Lambda \ll T_c$ one expects that the theory becomes scale invariant. This is what indeed happens, as can be seen by rescaling V_{Λ} and φ according to $V_{\Lambda} = \Lambda^3 T_c \, \tilde{V}_{\Lambda}$ and $\varphi = \sqrt{\Lambda T_c}\, \tilde{\varphi}$ (notice that the temperature enters as expected by dimensional reduction

arguments). In terms of the new variables, eq. (53) becomes

$$\frac{\partial}{\partial t}\tilde{V}_\Lambda + \tfrac{1}{2}\tilde{\varphi}\tilde{V}'_\Lambda - 3\tilde{V}_\Lambda = \frac{1}{2\pi^2}\log\left[1-\exp\left(-e^{-t}\sqrt{1+\tilde{V}''_\Lambda}\right)\right], \qquad t=\log\left(\frac{T_c}{\Lambda}\right).$$

As $t \to \infty$ the explicit t-dependence in the RHS disappears, and we obtain the scale-invariant equation (dropping tildes and changing variables to absorb the factor in front of the logarithm)

$$\frac{\partial}{\partial t}V_\Lambda + \tfrac{1}{2}\varphi V'_\Lambda - 3V_\Lambda = \log\left(1+V''_\Lambda\right). \tag{63}$$

This is the RG flow equation for the zero temperature potential in 3 space-time dimensions at the lowest order in the derivative expansion and for a sharp momentum cutoff [27]. We see than that the critical behaviour of (53) is effectively described by a three dimensional $T = 0$ theory. The fixed points and the corresponding critical indices of eq. (63) can be computed numerically without truncations and the result for ν is (see [27] for details)

$$\nu = 0.6895\,.$$

On the other hand, if one is interested in studying the theory out of the critical regime, or in the case of a first order phase transition, the three-dimensional approach is no longer suited, and the full four-dimensional theory has to be addressed. In this case, as we have discussed at the end of sect. 3, the present formulation of the Real Time RG provides a clear connection between the theory at finite temperature and the theory at $T = 0$, which we are supposed to test in the laboratory.

6 Conclusions and outlook

In thermal field theory several difficulties are encountered while performing perturbative calculations. These problems come from the severe infra-red divergences which plague finite temperature Green's functions, in particular in gauge theories. To this aim a technique for the resummation of the so-called "hard thermal loops" was proposed and developed [28]. However, this method is not successful for scales in which non-perturbative physics (such as the gluon magnetic mass in QCD) become relevant. The key problem is that ordinary perturbation theory does not clearly separate the various scales in the game, and the diagrammatic expansion has to be reorganized.

We believe that the Wilson RG approach should be very helpful in this sense.

In this paper we have considered the Wilson RG formulation of a RT thermal field theory. The main idea is to consider the thermal interaction between a quantum field and a thermal bath as an effective interaction, namely to regard the frequency modes above a certain scale Λ as effective interactions for the low energy modes below Λ. Therefore the thermal fluctuations above Λ are integrated out and put in a "Wilsonian effective action", which satisfies an "exact" (in principle) evolution equation in Λ. We solved this flow equation for the effective potential in various approximations, providing a non-perturbative resummation of Feynman graphs. The approach is physically quite transparent, rigorous, and gives better numerical results with respect to usual resummed perturbation theory.

The Wilson method had already been applied to the same self-interacting scalar model in the Imaginary Time Matsubara formalism of thermal field theories [16, 17]. Our results agree with these previous analyses and are better in some cases. However, the main aim of this paper was to set up the general formalism and discuss some approximation methods, by using the well-studied scalar theory.

Concerning the extension of this RT approach to gauge theories, we believe it is very promising, since the issue of gauge invariance can be considered in a clean way by using the CTP formalism described in the Appendix A. Moreover, if one is interested in the static quantities (such as effective potentials and thermal masses), the imaginary parts of the self-energies can be neglected and only on-shell modes contribute to the flow, as shown by (45). In this case, by imposing the thermal boundary conditions only on the physical degrees of freedom, as suggested in [29], one finds that the thermal on-shell part of the propagators is gauge-invariant. Therefore the RG evolution equations obtained in this way provide a gauge-invariant resummation. The consequences of this observation will be considered in a separate paper.

The RT formalism is absolutely necessary in order to deal with non-equilibrium phenomena. We hope this paper is a first step towards a consistent formulation of coarse-graining in non-equilibrium field theory. As far as the problem of thermalization is concerned, it should be possible to derive a Boltzmann equation, at least for a quasi-equilibrium or quasi-stationary system, and give a precise physical meaning to the IR cutoff Λ, which could be interpreted as the thermalization scale, $\Lambda(t)$, to be determined dynamically. With such a systematic treatment, one should be able to compute corrections to the Boltzmann equation, at least in some regimes. More generally, the dynamical elimination of the degrees of freedom which are irrelevant at a certain scale, would permit to derive an effective theory (for instance a kinetic theory [30]) from the underlying fundamental field theory. These points are currently being explored.

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A The Closed Time Path formalism

In this Appendix, we give a short review of the CTP formalism [4], and discuss its application to the formulation of the Wilson RG presented in this paper. We will follow the presentation of refs. [5] and [6] (see also the review [31]).

This technique was introduced to describe the evolution of an arbitrary initial state, described by a density matrix ρ.

In the usual path integral approach to scattering theory, we are interested to the transition amplitudes between a $|in\rangle$ and a $|out\rangle$ vacuum state in the presence of an external source. We now face a different physical problem. We want to follow the out-of-equilibrium time evolution of an interacting field, without knowing its state in the far future. Moreover, we look for a real and causal evolution equation. For a Hermitian field operator, the diagonal matrix element $\langle in|in\rangle$ is real and the off-diagonal ones $\langle out|in\rangle$ are complex.

Therefore the basic idea of the CTP method is to start from a diagonal $\langle in|in\rangle$ matrix element at a given time $t = 0$ and insert a complete set of states at a later time t':

$$Z[J_+, J_-] = \int [d\psi] \langle in, 0|\psi, t'\rangle_{J_-} \langle \psi, t'|in, 0\rangle_{J_+} \ , \tag{64}$$

where J_+ and J_- are the two sources for the propagation forward and backward in time, respectively. Notice that one puts different sources for the two transition amplitudes, since one is mainly interested in following the forward time evolution. By introducing the density matrix ρ of the initial state and setting $t' = +\infty$ one gets

$$\begin{aligned} Z[J_+, J_-, \rho] = \int [d\varphi][d\varphi'][d\psi] \langle \varphi, 0|T^* \exp\left(-i \int J_- \Phi\right) |\psi, +\infty\rangle \\ \times \langle \psi, +\infty|T \exp\left(i \int J_+ \Phi\right) |\varphi', 0\rangle \langle \varphi', 0|\rho|\varphi, 0\rangle \ , \end{aligned}$$

where T and T^* are the time and anti-time ordering operators, respectively. We see the origin of the name CTP. This path-integral is performed on a contour in the complex time plane running from $t = 0$ to $t = \infty$ and then back to $t = 0$. One can then introduce a standard path integral representation for the two matrix elements corresponding to the transitions from the state at $t = 0$ to the state at $t = t'$ and viceversa:

$$Z[J_+, J_-, \rho] = \int [d\phi_+][d\phi_-] \langle \phi_+, 0|\rho|\phi_-, 0\rangle \exp i \left\{ S[\phi_+] + J_+\phi_+ - S^*[\phi_-] - J_-\phi_- \right\} \ . \tag{65}$$

It is clear how the $+, -$ fields and sources correspond to the ones labelled with $1, 2$ in the paper, respectively. Therefore we introduce the vectors

$$\phi_a = (\phi_+, \phi_-) \ , \qquad J_a = (J_+, J_-) \ . \tag{66}$$

As far as the initial state is concerned, one can always decompose the density matrix element at $t = 0$ in the following way:

$$\langle \phi_1, 0|\rho|\phi_2, 0\rangle = \exp i \left\{ K + \int K_a \phi_a + \tfrac{1}{2} \int K_{ab} \phi_a \phi_b + \dots \right\} \ . \tag{67}$$

Therefore the information on the initial state is all contained in the non-local sources K, which are obviously concentrated at $t = 0$. The various K's are just boundary conditions in time for the corresponding Green functions. In this way a perturbative expansion may be constructed, with the usual Feynman rules.

We now specify the form of the initial state. For an initial state of a real free particle in thermal equilibrium, $\rho \sim \exp[-\beta \int d^3k \, \omega_k a_k^\dagger a_k]$ and the matrix element is [5]:

$$\langle \phi_1, 0|\rho|\phi_2, 0\rangle \sim \exp -\tfrac{1}{2} \int \frac{d^3k}{(2\pi)^3} \frac{\omega_k}{\sinh \beta\omega_k} [(\phi_1^2(k) + \phi_2^2(k)) \cosh(\beta\omega_k) - 2\phi_1(k)\phi_2(k)] \ . \tag{68}$$

We see that the effect of this initial state is just to add a correction to the free propagators of the theory. In this way the free propagator of the RT thermal field theory is obtained. For a interacting particle, the matrix element in (67) cannot be computed exactly, but the Wick theorem may be used to perform the usual perturbative calculations of the RT equilibrium theory. In this way we find the same result as we would have found by applying the well-known property that, for a pure thermal state, the matrix element $\langle \phi_1, 0|\rho|\phi_2, 0\rangle$ admits a path integral representation over the path going from $t = 0$ along the imaginary time axis to the time $t = -i\beta$. It may also be shown that indeed there is a certain freedom in the choice of the path to go from $t = 0$ to $t = -i\beta$, so that all the various formulations of the equilibrium thermal field theory are obtained.

The density matrix relevant to this paper, eq. (6), is obtained by introducing a momentum-dependent temperature

$$\beta_{k,\Lambda} = \begin{cases} \beta\omega_k & \text{for } |\vec{k}| > \Lambda \\ \\ +\infty & \text{for } |\vec{k}| < \Lambda \end{cases} . \tag{69}$$

As discussed in [5], if the density matrix has this form all the initial n-point correlations, K in (67), vanish for $n > 2$, and the result for the density matrix element (67) is obtained from eq. (68) after the substitution $\beta\omega_k \to \beta_{k,\Lambda}$.

As a consequence, the only modification with respect to the usual equilibrium RT formalism will consist in replacing the Bose-Einstein distribution function in the free propagators, with the cut off one defined in (11).

B Cancellation of pinch singularities

In this appendix we discuss the mechanism of cancellation of the pinch singularities which appear in the matrix kernel K_Λ (36), which we rewrite in the following way (we suppress the index Λ and restore the field indices 1,2 for sake of notation)

$$\frac{\partial}{\partial\Lambda}\Gamma_1^{(1)} = \tfrac{1}{2}\,\mathrm{tr}\, K_{ab}\Gamma_{1ab}^{(3)}\,. \tag{70}$$

We will be able to rewrite this equation in a way which is manifestly free of the pinch-like singularities $\Delta_0^m\Delta_0^{*n}$ appearing in eq. (40).

First of all recall that the full matrix propagator $G_\Lambda = [D_\Lambda^{-1} + \Sigma_\Lambda]^{-1}$ is free of pinch singularities due to the matrix structure of D_Λ and Σ_Λ [7], and so it is its Λ-derivative. Then the origin of the pinch singularities in the kernel of the RG flow equation (70) is clear if one realizes that the definition (36) corresponds to the Λ-derivative of the full propagator taking into account only the Λ-dependence coming from the *free* propagator D_Λ. This destroys the matrix structure of G_Λ. In order to recover it (and therefore to cancel the pinch singularities), from the vertex $\Gamma_{1ab}^{(3)}$ in (70) there should come a contribution like

$$-G_\Lambda \cdot \left(\frac{\partial}{\partial\Lambda}\Sigma_\Lambda\right) \cdot G_\Lambda\,,$$

which, added to the kernel (36), would give the total (pinch-free) Λ-derivative of G_Λ.

Indeed this happens in the following way. The Λ-derivative of the matrix self-energy is found by derivating with respect to ϕ_i, ϕ_j the evolution equation (32)

$$\frac{\partial}{\partial\Lambda}\Sigma_{ij} = \tfrac{1}{2}\,\mathrm{tr}\,K_{ab}[\Gamma^{(4)}_{ijab} - 2\Gamma^{(3)}_{ica}G_{cd}\Gamma^{(3)}_{jdb}]\,. \tag{71}$$

Then one has the identity

$$i\Gamma^{(3)}_{1ij} = i\tilde{\Gamma}^{(3)}_{1,ij} + \tilde{\Gamma}^{(3)}_{1,ab}G_{aa'}G_{bb'}[\tfrac{1}{2}\Gamma^{(4)}_{a'b'ij} - \Gamma^{(3)}_{a'ci}G_{cd}\Gamma^{(3)}_{b'dj}]\,, \tag{72}$$

where $\tilde{\Gamma}^{(3)}_{k,ij}$ is a vertex which is *two-particle-irreducible* if one tries to separate the external fields ij from the external field k. By inserting this equation in (70) and using (71) one has

$$\frac{\partial}{\partial\Lambda}\Gamma^{(1)}_1 = \tfrac{1}{2}\,\mathrm{tr}\left(i\frac{\partial}{\partial\Lambda}G_{ab}\right)\tilde{\Gamma}^{(3)}_{1,ab}\,,$$

which does not exhibit pinch singularities. We can summarize this mechanism of cancellation as follows: in the product $K_{ab}\Gamma^{(3)}_{1ab}$ the contributions corresponding to the Λ-derivative of the matrix self-energy come from the parts of $\Gamma^{(3)}_{1ij}$ in which the external fields ij are two-particle-reducible with respect to the external field 1 (the reader may convince himself of this by drawing the corresponding diagrams). These two-particle-reducible parts are the last term in the r.h.s. of eq. (72). Then the Λ-derivative of the matrix self-energy cooperates with the kernel K_Λ to give the Λ-derivative of the full propagator, which is free of pinch singularities.

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